

# Random Trees in Electrical networks

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## Abstract

This paper contains results relating currents and voltages in resistive networks to appropriate random trees or forests in those networks. Since each resistive network has a reversible Markov chain equivalent, we obtain equivalent results for the latter as well. We describe a way of obtaining a harmonic function on a weighted graph given the boundary values, by choosing random forests of the graph. As applications of the theorems discussed, (which give formulae of the Kirchhoff tree kind), we obtain an expression for the expected transit time from one state to another in a reversible Markov chain in terms of its arborescences.

The methods of this paper can also be used to give alternative proofs of the Kirchhoff tree formula.

## 1 Introduction

The first formula involving trees in electrical networks is due to Kirchhoff (see [1]). He gave an expression for the equivalent conductance between two nodes of a resistive (or as we shall henceforth say, conductive) network. Kirchoff's formula states that the equivalent conductance between two nodes of a network is  $\frac{\sum_i t_i}{\sum_j f_j}$ , where,  $t_i$  is the product of the conductances of the  $i$ th tree and  $f_j$  is the product of the conductances of the  $j$ th forest that separates the nodes in question.

In recent years, formulae involving trees have been discovered for general Markov chains too (see [2], [3]).

The physical model used to explain Ohm's law involves Brownian motions of charge carriers. So it is not surprising that a conductive network may be mapped onto a Markov chain with very similar properties. Trees seem to figure in Markov chains as "partial histories". In a Markov chain started at time  $-\infty$  and ended at time  $\tau$ , the set of directed edges corresponding to the final visits to each state, form a directed tree rooted at the state reached at time  $\tau$ . The evolution of these histories as the Markov chain progresses is another Markov chain which has arborescences (i.e rooted trees) as states. This Markov chain has interesting properties, and can be regarded as underlying the results that appear in this paper.

The formulae presented here are multiterminal extensions of the Kirchhoff tree formula. They all have equivalent forms in reversible Markov chains, which can be written out by simply substituting for injected current, branch current and node voltage, their Markov chain equivalents given in Section 2 .

We obtain Kirchhoff's formula for the equivalent conductance between two nodes, from the result we have called the VJ Theorem, when we apply a unit voltage across them. Since the proof of the VJ Theorem is purely graph theoretic, we get a new derivation of the Kirchhoff conductance formula as well, quite different in nature from the usual proof using Binet - Cauchy Theorem.

Although this is not the concern of this paper, we state that these formulae can be implemented in a natural way using randomized algorithms to give fast approximate

solutions to network problems. The essential idea is that we make numerous “observations” of the network, each of which is computationally light, and the “overall picture” that we get is an approximation to the network’s actual behaviour. The more observations we make, the better our approximation is expected to be. The critical step in each observation is that of picking a random tree (forest), the chance of picking a particular tree (forest) being proportional to the product of its branch conductances.

Picking random trees has been found useful in other contexts as well, such as optimizing on server positions in a computer network, and finding euler paths in a graph, and this problem has been well studied (see [4], [5], [10], [13]). There are a variety of efficient algorithms available that perform this task.

Here is a preview of three of the theorems in network form.

The VV Theorem (special case):

In a conductive network with nodes  $S_1, \dots, S_n$ , apply an external voltage source of 1 volt across  $S_1$  and  $S_2$  and choose  $S_2$  as the ground. Let  $v_k$  be the voltage at  $S_k$ . (Thus  $v_1 = 1, v_2 = 0$ ).

Then  $v_k$  is given as follows :

Consider all those forests  $f$  (of the network) which have  $n - 2$  branches, (consisting of two disjoint trees, whose union touches every vertex of the network), such that  $S_1$  and  $S_2$  are not vertices of the same tree in  $f$ .

Call this set  $F_{12}$ . Choose a forest randomly out of  $F_{12}$ , with probability proportional to the product of its branch conductances. The voltage  $v_k$  at node  $S_k$  is the probability that in the chosen forest  $f$ ,  $S_k$  is a node of the same tree as  $S_1$ .

This theorem is really a result on harmonic functions on weighted graphs, since but for the “poles”  $S_1$  and  $S_2$ , the voltage at any vertex is the weighted mean of its neighbour’s voltages.

The JI Theorem:

In the network mentioned above, let currents  $J_1, \dots, J_n$  be injected externally into nodes  $S_1, \dots, S_n$ . Choose a random spanning tree  $t$  of the network, the probability of choosing  $t$  being proportional to the product of its branch conductances. Setting all conductances other than those of  $t$  to 0, we get a certain current distribution according to which the current in any branch not in  $t$  is 0. The actual current distribution is the expected distribution, under a random choice of tree  $t$ , or the average of the distributions taken over all trees (the weight of a distribution being the product of the conductances of the corresponding tree).

The IV Theorem:

Let the  $J_k$ ’s be as above. Suppose we are given any current distribution in the branches of the graph that is consistent with the injected currents, but which does not necessarily satisfy the Kirchhoff voltage law. Choose a random tree  $t$  as in the JI Theorem. Take any vertex as ground (potential zero). Calculate the node potential of a vertex by adding the branch voltages along the unique path in  $t$  from the ground to that vertex. A correct node voltage distribution is given by the expected value of node potentials under the choice of a random tree  $t$ .

The JI and IV theorems above, involve choosing random trees while the VV Theorem involves choosing a random forest. In each case, the probability of a choice is proportional to the product of its conductances. There is a well-known algorithm using Markov-chains for choosing random trees with this probability distribution (see [4], [5]). As we shall see in the section with Theorem VV, this algorithm can be modified to choose random forests too.

## 2 The network - Markov equivalence

We now talk about the correspondence between electrical networks and reversible Markov chains. This is well-known (see [6], [7], [8], [9]), but has been included for the sake of completeness.

Consider a conductive network with nodes  $S_1, \dots, S_n$ . Let  $g_{kj} = g_{jk}$  be the branch conductance between the nodes  $S_j$  and  $S_k$ . If  $j = k$  the conductance obviously has no effect on the electrical behavior of the network, however, because in our Markov analogue they do contribute, we allow  $g_{jj}$  to be positive. Let  $v_k$  be the voltage of  $S_k$ ,  $J_k$  the current injected from outside the network into the node  $S_k$  and  $i_{kl}$  the branch current flowing from  $S_k$  to  $S_l$  in the conductance  $g_{kl}$ . (Conductance is the inverse of resistance. In this paper, where there is no ambiguity, we shall use the word conductance for a branch as well.)

The Kirchhoff current law is expressed in the following equation :-

$J_k = \sum_{m=1}^n g_{mk} (v_k - v_m)$ . Let the injected current vector be called  $J$ , the vector of node voltages be  $V$ , and the branch current matrix be  $I$ . Thus  $J = (J_1, \dots, J_n)$ ,  $V = (v_1, \dots, v_n)$ , and  $I = \{i_{kl}\}_{n \times n}$ .

Consider the Markov chain with states  $S_1, \dots, S_n$  (these will correspond to nodes of our network anyway, so there is no clash of notations). Let transition probability  $p_{kl}$  be  $\frac{g_{kl}}{g_k}$  where  $g_k = \sum_{m=1}^n g_{km}$ . Let us further assume that

$$\sum_{k=1}^n \sum_{l=1}^n g_{kl} = 1.$$

This is just a scaling of the conductances, but is convenient. We assume that the network is connected, and therefore that the corresponding Markov chain has a path of positive probability from any state to any other - i.e the markov chain is strongly connected. If it is also aperiodic, it has a unique stationary distribution, which can be easily verified to be  $\pi = (g_1, \dots, g_n)$ , with the scaled conductances. Let the initial probability distribution be  $p^{(0)}$ . Suppose we have a stopping rule under which the expected run-time, for some (and hence, since the Markov chain is strongly connected and finite, every) initial probability distribution is finite. Let the probability of the walk terminating at  $S_k$  be  $p_k^{(u)}$ , and the corresponding vector be  $p^{(u)}$ . Let  $e_k$  be the expected number of times the walker visits state  $S_k$ , where the last move of the walk at which the walker stops is not counted as a visit. Thus for example if the walk began at  $S_1$  and the stopping rule declares that at the first visit to  $S_k$ , the walk ends, then according to us the number of visits to  $S_k$  is always 0. With this notation, the following statement can be easily verified for each  $k$ :

$$p_k^{(0)} - p_k^{(u)} = e_k - \sum_{m=1}^n (p_{mk} e_m).$$

This essentially says that “initial probability - final probability = net outflow - net inflow” In our present situation,  $p_{mk} = \frac{g_{mk}}{g_m}$  so we have

$$p_k^{(0)} - p_k^{(u)} = e_k - \sum_{m=1}^n \left( \frac{g_{mk}}{g_m} \right) e_m.$$

In anticipation, let  $\frac{e_m}{g_m}$  be called  $v_m$  (this will actually behave like voltage), and let  $J_k$  denote the L.H.S (which will play the role of injected current). Our equation then becomes  $J_k = g_k \times v_k - \sum_{m=1}^n (g_{mk} v_m)$  or  $J_k = \sum_{m=1}^n g_{mk} (v_k - v_m)$ ,

which is identical to the Kirchhoff current equation we had earlier. This verifies the correspondence. In a network,  $i_{kl} = (v_k - v_l)g_{kl}$ , which therefore becomes  $e_k(p_{kl}) - e_l(p_{lk})$  in the Markov chain. Therefore the Markov analogue of current  $i_{kl}$  is the expected difference between the number of transitions from  $S_k$  to  $S_l$  and the number of transitions from  $S_l$  to  $S_k$ . In the network results that follow, whenever we use the phrase “A node  $S_k$  whose voltage is fixed externally”, we mean that the injected current  $J_k$  at  $S_k$  is not necessarily 0. However, for a node  $S_l$  whose voltage is not fixed externally,  $J_l$  is necessarily 0.

### 3 Preliminaries

An arborescence of a directed graph is a tree, in which a node has been singled out as root and all branches are so directed that from any node of the graph to the root, there is a unique directed path. A Markov chain is a process consisting of a succession of events where the probability of an event happening is a function of the preceding event that has occurred in the process. In this paper, we only consider Markov chains which are finite and which are strongly connected. By strongly connected, we mean that if  $E_1$  and  $E_2$  are any events in our sample space, and our process launches itself with  $E_1$ , the probability that  $E_2$  occurs  $n$  events later is non-zero for some integer  $n$ . If a Markov chain is aperiodic and strongly connected, it is a result that whatever be the probability distribution with which the process begins, the probabilities of the various events tend to a “stationary probability distribution” if we wait sufficiently long. We call the events from the sample space of a Markov chain, states, and given that  $E_i$  has just occurred, call the probability that  $E_j$  occurs next, the transition probability from  $E_i$  to  $E_j$ , which we denote by  $p_{ij}$ . Let the stationary probability distribution of a finite, irreducible Markov chain be  $\{\pi_1, \pi_2, \dots\}$ , corresponding to events  $\{E_1, E_2, \dots\}$ . If  $\pi_i \times p_{ij} = \pi_j \times p_{ji}$ , the Markov chain is called reversible.

Let  $R$  be a subset of  $S = \{S_1, \dots, S_n\}$ . We then call by  $F_R$  the set of all maximal forests of the network that “separate” states in  $R$ . Thus  $F_R$  consists of all those subgraphs  $f$  of the network for which any  $S_k$  in  $S$  is connected by a unique (conducting) path in  $f$  to some state in  $R$  (a state is always regarded to be connected to itself by the null path). It follows that by the condition of connectivity in  $f$ , the states of  $S$  are partitioned into  $|R|$  blocks such that each block has exactly one state of  $R$ . If  $S_k$  belongs to  $R$  and  $f$  to  $F_R$ , we shall denote by  $B_f(k)$ , the set of states which are connected to  $S_k$  by a path in  $f$ . We will often perform algebraic operations with a forest  $f$  in  $F_R$ . In every such case,  $f$  is interpreted as the product of the conductances of the forest  $f$ . This is a helpful abuse of notation; it creates no ambiguity, but does simplify our expressions. (For example,  $f_1 + f_2$  represents the sum of the products of conductances of the forests  $f_1$  and  $f_2$ .)

We define  $S_f(k)$ , for  $f$  in  $F_R$  to be the unique state (we henceforth use ‘state’ synonymously with node and vertex, since nodes become states of the related Markov chain) of  $R$  that  $S_k$  is connected to by a path in  $f$ .  $v_f(k)$  is taken to be the voltage of the state  $S_f(k)$ . It is true that a forest  $f$  may belong to both  $F_R$  and  $F_Q$ , where  $R$  and  $Q$  are different vertex sets, but in all our expressions, we talk of forests  $f$  in a particular  $F_R$ , and so  $S_f(k)$ , and  $v_f(k)$  are well defined by the context in which they appear.

## 4 The VJ Theorem

We now give a formula for the injected currents  $J_k$  in terms of the externally fixed voltages of the network. When  $J$  is the vector  $(-1, 1, 0, \dots, 0)$ , we get Kirchhoff's formula for equivalent conductance.

Theorem VJ:

Let  $R$  be a nonempty subset of  $S$  not containing  $S_1$ . Let  $Q = R \cup S_1$ . Let the voltages at nodes of  $Q$  be the only ones fixed externally.

Then

$$J_1 = \frac{\sum_{h \in F_R} (v_1 - v_h(1)) h}{\sum_{f \in F_Q} (f)}.$$

Proof :

We say that a forest  $f_1$  is contained in another forest  $f_2$  if every branch of  $f_1$  is also a branch of  $f_2$ . If  $f$  belongs to  $F_Q$ ,  $h$  belongs to  $F_R$  and  $f$  is contained in  $h$ , then  $h$  has exactly one branch that  $f$  doesn't, which we shall denote as  $h/f$ . If we look at the partition of  $S$  induced by the forest  $f$  (in which, we recall, every block has exactly one representative of  $Q$ ), we see that  $h/f$  has exactly one endpoint in  $B_f(1)$ . Let  $v(h/f)$  be the voltage of  $h/f$ , taken directed outward from  $B_f(1)$ . We observe that the net current leaving  $B_f(1)$ , (and entering states outside) is  $J_1$  for any  $f$  in  $F_Q$  since the only state in  $B_f(1)$  that can possibly have non-zero injected current is  $S_1$ . Therefore,

$$J_1 = \sum_{(k, l \text{ s.t. } S_f(k) = S_1 \neq S_f(l))} g_{kl} (v_k - v_l).$$

i.e

$$\begin{aligned} J_1 \left( \sum_{f \in F_Q} f \right) &= \\ \sum_{f \in F_Q} \left( \sum_{(k, l \text{ s.t. } S_f(k) = S_1 \neq S_f(l))} (f) (g_{kl}) (v_k - v_l) \right). \end{aligned}$$

This may be rewritten as

$$\begin{aligned} &\sum_{(f \in F_Q, h \in F_R \text{ s.t. } f \subset h)} h \times v(h/f) \\ &= \sum_{h \in F_R} h \times \left( \sum_{(f \in F_Q \text{ s.t. } f \subset h)} v(h/f) \right) \end{aligned}$$

But for a fixed  $h$  in  $F_R$ ,

$$\sum_{(f \in F_Q \text{ s.t. } f \subset h)} v(h/f)$$

is the sum of branch voltages along the path from  $S_1$  to  $S_h(1)$  contained in  $h$ , and this is just  $v_1 - v_h(1)$ . Therefore

$$J_1 \left( \sum_{f \in F_Q} f \right) = \sum_{h \in F_R} h \times (v_1 - v_h(1)).$$

This proves the theorem.

The VJ Theorem has a probabilistic interpretation. In what follows, whenever we say choose a random forest in  $F_Q$ , it is implicit that the probability of choosing

$f$  is proportional to the product of its conductances. Similarly by a random branch we mean that it is chosen with probability proportional to its conductance. Let  $g$  be the sum of all conductances of our network. Choose a random forest  $f$  in  $F_Q$ . Choose a random branch  $g_{kl}$  from the network. Then  $S_f(k)$  and  $S_f(l)$  are states in  $Q$ . Let the number of branches in the unique path from  $S_m$  to  $S_f(m)$  in  $f$  be called  $d_f(m)$  for each  $m$  and each  $f$ .

Consider the current vector  $J(f, kl)$  given by

$$J_m = 0 \text{ if } S_m \text{ is not } S_f(k) \text{ or } S_f(l).$$

$$J_m = \frac{g \times (v_f(k) - v_f(l))}{1 + d_f(k) + d_f(l)} \text{ if } m = S_f(k).$$

$$J_m = \frac{g \times (v_f(l) - v_f(m))}{1 + d_f(k) + d_f(l)} \text{ if } m = S_f(l)$$

Then, the theorem says that the expected value of  $J(f, kl)$  when  $f$  and  $g_{kl}$  are random, is the actual  $J$  that is injected into the network.  $1 + d_f(k) + d_f(l)$  is just the length of the connecting path via  $g_{kl}$  ( in  $f$  ) between  $S_f(k)$  and  $S_f(l)$  when these are distinct ; it was used to simplify the expression.

To see why the two forms of the VJ Theorem are identical, let us find the expected value of  $J_1(f, kl)$ , the first component of  $J(f, kl)$ . If  $S_1$  is not in  $Q$ , the result is obvious since then  $J_1(f, kl)$  is always 0, so we assume that  $S_1$  belongs to  $Q$  and let  $R = Q - S_1$ .

$$E[J_1(f, kl)] = \frac{\sum_{f \in F_Q} \left( \sum_{(k, l) \text{ s.t. } S_k \in B_f(1)} (f) \left( \frac{g_{kl}}{g} \right) (v_1 - v_f(l)) \left( \frac{g}{1 + d_f(k) + d_f(l)} \right) \right)}{\sum_{f \in F_Q} f}.$$

We observe that if  $S_l$  is not in  $B_f(1)$ , then  $f \times g_{kl}$  corresponds to a forest  $h$  in  $F_R$ , while if it is,  $v_1 - v_f(1)$  is 0. Further, the number of pairs of the form  $(f, g_{kl})$  that lead to a forest  $h$  in  $F_R$ , is the number of branches in the unique path from  $S_1$  to  $S_h(1)$  in  $h$ . This is the same as  $1 + d_f(k) + d_f(l)$  for any pair  $\{f, g_{kl}\}$  that gives rise to  $h$ . Therefore the above expression is equal to

$$\frac{\sum_{h \in F_R} (v_1 - v_h(1)) \times h}{\sum_{f \in F_Q} f},$$

which is  $J_1$ .

## 5 The VV Theorem

Theorem VV :

Let  $R$  be a non-empty subset of  $S$ , not containing  $S_1$ . Let the states of  $R$  be the only ones whose voltages are fixed externally, (i.e for all  $S_l$  in  $S - R$ , let  $J_l$  be 0). Then,

$$v_1 = \frac{\sum_{h \in F_R} (v_h(1) \times h)}{\sum_{h \in F_R} (h)}$$

Proof :

Let  $Q = R \cup S_1$ . By the VJ Theorem,

$$J_1 = \frac{\sum_{h \in F_R} (v_1 - v_h(1)) \times h}{\sum_{f \in F_Q} f}.$$

$J_1 = 0$ , and so

$$0 = \sum_{h \in F_R} ((v_1 - v_h(1)) \times h),$$

which implies the stated theorem.

All that was needed in proving the above VV Theorem is that for any state  $S_l$  in  $S - R$ ,

$$v_l = \frac{\sum_{m=1}^n ((g_{lm})(v_m))}{\sum_{m=1}^n g_{lm}}$$

holds, which is a condition of harmonicity outside  $R$ .

Here is the equivalent probabilistic version :

Choose a random forest  $h$  out of the set  $F_R$ , with probability proportional to the product of its conductances. The voltage at  $S_1$ , then is the expected value of the voltage of the unique state of  $R$  that  $S_1$  is connected to via  $h$ .

A Markov chain, having elements of  $F_R$  as states and stationary probability of  $f$  proportional to the product of its conductances can be obtained from the Markov chain that gives random trees. Take the network, and fuse all states of  $R$ . We now have a new conductive network whose branches all come from the original (though some may have become parallel). It is clear that the forests in  $F_R$ , consist of precisely those collections of branches that form the trees of the new network, and that this correspondence is bijective. Now, simply use the tree Markov chain to give trees of the new network (this method works perfectly even when we have parallel branches), and pick corresponding forests from  $F_R$ . This gives random forests of  $F_R$  with the right probabilities.

## 6 The JI Theorem

Here is a theorem that expresses the current distribution in a network in terms of the injected currents. Let  $T$  be the set of all (spanning) trees. Let  $I$  be the current matrix  $\{i_{kl}\}_{n \times n}$  where  $i_{kl}$  is the current flowing in the conductance from  $k$  to  $l$ . Let  $J$  be the vector of injected currents. If we were to set all conductances other than those in a particular tree  $t$  to 0, we would have a current distribution that would be 0 in all branches except those of  $t$ . Let this current distribution in matrix form be denoted as  $I_t$ .

Theorem JI :

$$I = \frac{\sum_{t \in T} t \times (I_t)}{\sum_{t \in T} t}$$

(where as in our previous theorems, when  $t$  appears in an algebraic expression, it is taken to be the product of the conductances in  $t$ .)

Proof :

The current matrix  $I$  is a linear function of the vector of injected currents  $J$ . A vector is a valid  $J$  if and only if the sum of its components  $J_1, \dots, J_n$  is 0. Such vectors form a vectorspace and it is sufficient to prove the theorem for a basis of this vectorspace. It is clear that vectors of the form  $z_k$ ,  $k = 2$  to  $n$  form a basis, where  $z_k$  is the vector with  $(z_k)_1 = -1$ ,  $(z_k)_k = 1$  and all other entries  $(z_k)_l = 0$ . For simplicity in notation, we shall prove the theorem for  $J = z_2$ . Let  $S_1$  be taken to be ground, ( i.e  $v_1 = 0$  ). Let  $R$  be  $\{S_1, S_2\}$ . We need to prove that

$$i_{lk} = \frac{\sum_{t \in T} (I_t)_{lk} \times t}{\sum_{t \in T} t}$$

for each pair  $\{S_l, S_k\}$ . We know that  $i_{lk} = g_{lk} \times (v_l - v_k)$  which, using the VV Theorem is

$$g_{lk} \times \frac{(\sum_{h \in F_R} h \times v_h(l)) - (\sum_{h \in F_R} h \times v_h(k))}{\sum_{h \in F_R} h}.$$

This is

$$\frac{g_{lk} \sum_{h \in F_R} h(v_h(l) - v_h(k))}{\sum_{h \in F_R} h}.$$

Now, the possible values of  $v_h(l) - v_h(k)$  over varying forests  $h$  are  $-v_2, v_2$  and 0, which arise respectively from the cases when

- (a)  $S_l$  is in  $B_h(1)$  and  $S_k$  is in  $B_h(2)$
- (b)  $S_l$  is in  $B_h(2)$  and  $S_k$  is in  $B_h(1)$
- (c) Both belong to the same block with respect to  $h$ .

Therefore,

$$i_{lk} = \left( \frac{g_{lk}}{\sum_{h \in F_R} h} \right) \times \left\{ \sum_{(h \in F_R \text{ s.t. } S_l \in B_h(2) \text{ and } S_k \in B_h(1))} h \times v_2 + \sum_{(h \in F_R, S_l \in B_h(1), \text{ s.t. } S_k \in B_h(2))} h \times (-v_2) \right\}.$$

From Theorem VJ,

$$J_2 = \frac{\sum_{t \in T} (t) \times (v_2 - v_1)}{\sum_{h \in F_R} h}.$$

So,

$$1 = \frac{\sum_{t \in T} t \times (v_2)}{\sum_{h \in F_R} h},$$

and therefore

$$v_2 = \frac{\sum_{h \in F_R} h}{\sum_{t \in T} t}.$$

This is actually just Kirchhoff's formula for equivalent resistance. We have it as a special case of Theorem VJ. Substituting this for  $v_2$ ,

$$i_{lk} = \left( \frac{g_{lk}}{\sum_{t \in T} t} \right) \times \left( \sum_{(h \in F_R \text{ s.t. } S_l \in B_h(2) \text{ and } S_k \in B_h(1))} h + \sum_{(h \in F_R \text{ s.t. } S_l \in B_h(1) \text{ and } S_k \in B_h(2))} -h \right)$$

Given a tree  $t$ , if  $g_{lk}$  is in  $t$ , call by  $\bar{h}$  the forest corresponding to  $\frac{t}{g_{lk}}$  (which therefore "separates"  $S_l$  and  $S_k$ ). Then,  $(I_t)_{lk}$  is

- (a) 0 if  $\bar{h}$  is not in  $F_R$ .
- (b) 1 if  $\bar{h}$  is in  $F_R$ ,  $S_{\bar{h}}(l) = S_2$  and  $S_{\bar{h}}(k) = S_1$ .
- (c) -1 if  $\bar{h}$  is in  $F_R$ ,  $S_{\bar{h}}(l) = S_1$  and  $S_{\bar{h}}(k) = S_2$ .

These exhaust all possibilities. Therefore,

$$\frac{\sum_{t \in T} (I_t)_{lk} \times t}{\sum_{t \in T} (t)} = \left( \frac{g_{lk}}{\sum_{t \in T} t} \right) \times \left\{ \sum_{(h \in F_R \text{ s.t. } S_l \in B_h(2) \text{ and } S_k \in B_h(1))} (h) + \sum_{(h \in F_R \text{ s.t. } S_l \in B_h(1) \text{ and } S_k \in B_h(2))} (-h) \right\},$$

which from what we just saw is  $i_{lk}$ . This proves our theorem.

The JI Theorem has a compact probabilistic interpretation. Choose a random tree  $t$  (with probability proportional to  $t$ ), and calculate the branch current matrix after all conductances not in  $t$  have been set to 0. Then, the expected value of the current distribution that we would get is the actual distribution.

## 7 The IV Theorem

This theorem gives node voltages for a particular  $J$ . Like the previous theorem, it involves picking a random tree. Let  $J$  be the vector of injected currents. Let  $I$  be a current matrix  $\{i_{lm}\}_{n \times n}$  such that  $\sum_{m=1}^n (i_{lm}) = J_l$ .

We note that this need not be the actual current distribution that results from  $J$  in the network we are working with. We could, for example, obtain a valid  $I$  of this kind by taking some tree of the network and finding the current distribution if all other conductances were set to 0. This is (in general) a much easier task than solving for the branch currents in our network which could have as many as  $\frac{(n)(n-1)}{2}$  branches.

Theorem IV :

Choose a random tree  $t$  in  $T$  (with notation from Theorem JI), with probability proportional to the product of its conductances. Choose  $S_1$  as ground (this choice is arbitrary). For each  $S_k$ , walk along the unique path that is in tree  $t$  from  $S_1$  to  $S_k$ , find

$$\sum_{\text{all } g_{lm} \text{ traversed from } l \text{ to } m \text{ in that path}} \left( \frac{i_{lm}}{g_{lm}} \right),$$

call this  $v_t(k)$ , and call the voltage vector  $(0, v_t(2), \dots, v_t(n))$   $V_t$ .

Then, the actual voltage vector  $V$  (with  $S_1$  as ground) is the expected value of the voltage vectors  $V_t$ .

i.e

$$V = \frac{\sum_{t \in T} t \times (V_t)}{\sum_{t \in T} t}$$

Proof:

Consider the  $\frac{n(n+1)}{2}$  current matrices  $I[k, l]$  corresponding to ordered pairs  $(k, l)$ ,  $k > l$ , such that in  $I[k, l]$ ,  $i_{kl} = 1$ ,  $i_{lk} = -1$  and all other entries are 0. These form a basis of the vectorspace of all possible current matrices. Given a current matrix  $I$ , there is a unique injected current vector  $J$  corresponding to it which is a linear function of  $I$ . Given  $J$ , and the values of the conductances, taking  $S_1$  as ground, there is a single (node) voltage vector  $V$ . All of these are related linearly, so it is enough to prove the theorem in the case where  $I$  has  $i_{21} = 1$ ,  $i_{12} = -1$ , and all other entries 0. If we prove this case, it proves the theorem for any of our basis matrices; the fact that  $S_1$  has been chosen as ground does not affect generality - it only causes a constant shift in the voltages. The  $I$  we have chosen corresponds to  $J = (-1, 1, 0, \dots, 0)$ . It clearly suffices to prove that the theorem gives the correct value of  $v_3$  and of  $v_2$ .

To prove the  $v_3$  case, we partition the set of all trees into three classes :

$T_0$ , the set of all trees  $t$  which do not contain the branch  $g_{12}$ .

$T_1$ , the set of all trees  $t$  which contain  $g_{12}$  and have the property that the path joining  $S_3$  and  $S_1$  in  $t$  does not go through  $S_2$ .

$T_2$ , the set of all trees  $t$  which contain  $g_{12}$  and have the property that the path joining  $S_3$  and  $S_1$  in  $t$  goes through  $S_2$ .

If we choose a random tree  $t$ ,

$(V_t)_3$  is 0 if  $t$  is in  $T_0$  or  $T_1$ . If  $t$  is in  $T_2$ ,  $(V_t)_3$  is  $\frac{1}{g_{12}}$ .

Let  $R = \{S_1, S_2\}$ . Then,  $t$  is in  $T_2$  if and only if  $g_{12}$  is in  $t$  and the forest  $h$  obtained by removing  $g_{12}$  from  $t$  (which always is in  $F_R$ ) is such that  $S_h(3) = S_2$ . Therefore,

$$\sum_{t \in T} (V_t)_3 t = \sum_{h \in F_R, S_h(3) = S_2} (1/g_{12}) \times (h \times g_{12}) = \sum_{h \in F_R, S_h(3) = S_2} h.$$

Using the VJ Theorem, we have proved that if  $J = (-1, 1, 0, \dots, 0)$ ,

$$\text{then } v_2 = \frac{\sum_{h \in F_R} h}{\sum_{t \in T} (t)}, \text{ where } S_1 \text{ is ground. This proves the theorem for } v_2. \text{ From this,}$$

$$\frac{1}{\sum_{t \in T} t} \times \sum_{h \in F_R \text{ s.t. } S_h(3) = S_2} (h) = \frac{v_2}{\sum_{h \in F_R} h} \times \sum_{h \in F_R, S_h(3) = S_2} h,$$

which from the VV Theorem is  $v_3$ . This completes our proof.

## 8 Applications to reversible Markov chains

Using the equivalence that was mentioned in the beginning, all of the network theorems can be translated quite literally into theorems for reversible Markov chains. Here however we shall only consider some interesting special cases. We start with a reversible Markov chain given by transition matrix  $P$  with states  $(S_1, \dots, S_n)$ . Construct the equivalent electrical network in which the nodes are the states  $S_k$  and the conductance  $g_{kl}$  is  $p_{kl}(\pi_k) = p_{lk}(\pi_l)$ ,  $\pi$  being the stationary distribution.

Then  $p_{kl} = \frac{g_{kl}}{g_k}$ , where

$$g_k = \sum_{m=1}^n g_{km}.$$

$g_k$  becomes the stationary probability at  $S_k$ . We will need to manipulate clumps of arborescences or “orchards” of the weighted directed graph represented by our Markov chain, so we need some more notation. (here the weight of the directed edge from  $k$  to  $l$  is taken as  $p_{kl}$ ). An orchard is a rooted forest; If we take a forest  $f$ , choose a root for every connected component of it, and direct all the edges in each connected component, so that there is a directed path from every state to the root of the component of  $f$  that it is in, we get an orchard. An orchard is fully determined by the forest that is its imprint in the graph, and the set of nodes or states that are its roots. The orchard corresponding to a forest  $f$ , and root set  $R$  will be denoted by  $[f, R]$ . We shall denote the product of its edge transition probabilities by  $o[f, R]$ . Let  $R$  be a subset of  $S$ . Assume for convenience that  $S_1$  is not in  $R$ , but  $S_2$  is.

Consider a random walk that originates at  $S_1$  at time 0. Impose a stopping rule according to which we stop the walk the first time the walker reaches a state in  $R$ . We shall now give formulae for the expected duration of the walk (which we shall henceforth call  $\tau_R$ ), and the probability that the walk terminates at  $S_2$ . The electrical equivalent of this problem is as follows (from what we did in Section 2): The voltages of states in  $Q = R \cup \{S_1\}$  are fixed externally, with the voltages of states in  $R$  set to 0. We don’t know  $v_1$ , (which is the expected number of visits to  $S_1$  into  $g_k$ ) but we do know that  $J_1 = 1$ , since the walker is known to start from  $S_1$  with probability 1. From the VJ Theorem, we have

$$1 = J_1 = \frac{\sum_{h \in F_R} (h)(v_1 - v_h(1))}{\sum_{f \in F_Q} (f)}.$$

In our problem,  $v_h(1)$  is always zero. So

$$v_1 = \frac{\sum_{f \in F_Q}(f)}{\sum_{h \in F_R}(h)}$$

Let  $e_k$  be the expected number of departures from the state  $S_k$  during the course of the walk.

$$\tau_R = \sum_{k=1}^n e_k = \sum_{k=1}^n v_k \times g_k.$$

Using Theorem VV, this becomes

$$\frac{\sum_{k=1}^n (g_k) \sum_{f \in F_Q} f \times v_f(k)}{\sum_{f \in F_Q}(f)} = \frac{\sum_{f \in F_Q} \sum_{k \text{ s.t. } S_k \in B_f(1)} (f \times g_k \times v_1)}{\sum_{f \in F_Q}(f)}.$$

$(v_f(k)$  is non-zero only if  $S_k$  is in  $B_f(1)$ ). Substituting for  $v_1$ , this is

$$\frac{\sum_{f \in F_Q} \sum_{k \text{ s.t. } S_k \in B_f(1)} (f \times g_k)}{\sum_{h \in F_R}(h)}.$$

Dividing numerator and denominator by  $\prod_{S_k \in S-R} (g_k)$ , we get

$$\tau_R = \frac{1}{\sum_{h \in F_R} o[h, R]} \times \sum_{f \in F_Q} \sum_{(k \text{ s.t. } S_k \in B_f(1))} o[f, R \cup \{S_k\}],$$

which is a formula only in terms of transition probabilities of the Markov chain. The probability that the chain terminates at  $S_2$  is the current that flows out of the network from  $S_2$ , which is  $-J_2$ . We use the VJ Theorem to find this. Let set  $U$  be  $Q - \{S_2\}$ .

$$\begin{aligned} -J_2 &= \frac{1}{\sum_{f \in F_Q} f} \times \sum_{u \in F_U} ((v_u(2) - v_2)(u)) \\ &= \frac{1}{\sum_{f \in F_Q} f} \times \sum_{u \in F_U, \text{ s.t. } S_u(2) = S_1} (v_1)(u) \\ &= \frac{1}{\sum_{h \in F_R} h} \times \sum_{u \in F_U, \text{ s.t. } S_u(2) = S_1} (u) \\ &= \frac{1}{\sum_{h \in F_R} h} \times \sum_{h \in F_R, \text{ s.t. } S_h(1) = S_2} (h) \end{aligned}$$

Dividing numerator and denominator by  $\prod_{S_k \in S-R} g_k$ , we have

$$-J_2 = \frac{1}{\sum_{h \in F_R} o[h, R]} \times \sum_{h \in F_R \text{ s.t. } S_1 \in B_h(2)} (o[h, R])$$

This is the probability that the walk terminates at  $S_2$ . We shall now give an application of Theorem JI. Suppose, in the Markov chain considered above, we started out with an initial probability distribution  $p^{(0)}$ . Let us denote by  $(i^{(m)})_{kl}$  the expected number of transitions from  $k$  to  $l$  minus those from  $l$  to  $k$  upto time  $m$ . Let  $I^{(m)}$  be  $\{(i^{(m)})_{kl}\}_{n \times n}$ . We shall find a formula for  $\lim_{m \rightarrow \infty} I^{(m)}$  which we call  $I^{(\infty)}$ . This is in some sense the net flow that takes place in the Markov chain to reach equilibrium. let  $p^{(m)}$  be the resulting probability distribution at time  $m$ . Let  $J^{(m)}$  be  $p^{(0)} - p^{(m)}$ . Then, by the equivalence seen earlier,  $I^{(m)} = \frac{\sum_{t \in T} I_t t}{\sum_{t \in T} t}$ , where  $t$  and  $T$  are from the corresponding electrical network, and have their customary meanings ( $I_t$  being calculated using  $J^{(m)}$ ). (We have this formula whenever the

expected run-time is finite.) Therefore, to find  $I$  in the limit, it is enough to use  $J^{(\infty)} = p^{(0)} - \pi$ , where  $\pi$  is the stationary probability, since  $p^{(m)}$  tends to  $\pi$  as  $m$  tends to infinity. With our scaling of conductances,  $\pi$  is simply  $(g_1, \dots, g_n)$ . Let vector  $w_k$  be a vector of length  $n$ , where  $(w_k)_l$ , the component in its  $l$ th position, is  $p^{(0)}$  if  $l \neq k$ , and  $p^{(0)} - 1$  if  $l = k$ . This clearly is a valid  $J$  vector, and further,  $J^{(\infty)} = \sum_{m=1}^n ((g_m)(w_m))$ . For an arborescence  $[t, S_m]$  (the orchard corresponding to a spanning tree rooted at  $S_m$ ), given a probability distribution  $p$ , we talk about the “flow”  $U([t, S_m])$  through it. This is an  $n \times n$  matrix whose  $kl$ th entry,  $u_{kl} = 0$  if neither  $p_{kl}$  nor  $p_{lk}$  is a directed edge of the arborescence.

If  $p_{kl}$  is in  $[t, S_m]$ ,  $u_{kl}$  is the sum of probabilities of all states for which the path to  $S_m$  in the arborescence goes through  $p_{kl}$ .

If  $p_{lk}$  is in  $[t, S_k]$ ,  $u_{kl}$  is  $-u_{lk}$ , where  $u_{lk}$  is defined using the previous statement. Let  $A$  be the set of all arborescences. If  $a \in A$ , let  $o(a)$  be the product of  $a$ ’s transition probabilities.

(All this seems a little elaborate and artificial, but is necessary if we want to get results about the reversible Markov chain from its own parameters, without going to the equivalent electrical network.) Our claim is that

$$I^{(\infty)} = \frac{\sum_{a \in A} (U(a) o(a))}{\sum_{a \in A} o(a)},$$

where the flow matrix  $U(a)$  is calculated using  $p^{(0)}$ . We observe that here,  $U([t, S_m])$  is just  $I_t$  calculated using  $w_m$  as the injected current vector. Also,  $o[t, S_k] = \frac{(t)(g_k)}{\prod_{m=1}^n g_m}$ , and therefore  $\frac{o[t, S_k]}{g_k} = \frac{o[t, S_l]}{g_l}$ . Thus,

$$\frac{\sum_{t \in T} (I_t t)}{\sum_{t \in T} t},$$

(where  $J$  is  $w_k$ )

$$\begin{aligned} &= \frac{\sum_{t \in T} (U([t, S_k]) o[t, S_k])}{g_k} \times \frac{\prod_{m=1}^n g_m}{\sum_{t \in T} (t)} \\ &= \frac{\sum_{t \in T} (U([t, S_k]) o[t, S_k])}{\sum_{t \in T} (\sum_{m=1}^n o[t, S_m])} \end{aligned}$$

(The denominator takes this form when we use  $\sum_{k=1}^n (g_k) = 1$ ) Now, using the fact that  $J^{(\infty)} = \sum_{m=1}^n (g_m w_m)$ , and combining linearly the corresponding expressions for  $I$ , we get the desired result.

## 9 Conclusion

In this paper we have proved four theorems about electrical networks, namely the theorems VJ, JI, IV and VV. Theorem VJ expresses the currents injected into the network in terms of the voltages of nodes into which they are injected. Theorem JI expresses branch currents in terms of the currents injected externally. In Theorem IV, an arbitrary current distribution is taken which, for the given injected currents, satisfies the Kirchoff current law. The voltages at nodes are expressed in terms of this current distribution. The VV Theorem expresses the voltages of each node in terms of the voltages of the nodes into which currents are injected. The mode of proof used for these results can also be used to give alternative proofs of the Kirchhoff tree formula. We have noted that the formulae give expressions in terms

of expected values, and so might lead to efficient methods of solving resistive circuits approximately. Finally, we have obtained three formulae for reversible Markov chains, one of which gives an expression for transit time from one state to another.

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